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Mixing and transport in stars – I. Formalism: momentum, heat and mean molecular weight

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ABSTRACT

The goal of this paper is to derive analytic expressions for the turbulent fluxes of momentum (Reynolds stresses), heat and mean molecular weight.

(i) Angular momentum. To solve the angular momentum equation one needs to know the Reynolds stresses R_{ij}, in particular $R_{\phi r}$. It is shown that the latter has the form $R_{r\phi} = -2D_s S_{\phi r} - 2D_v V_{\phi r} - D_0\Omega_0 - D_1\Omega + \ldots$, where $2S_{\phi r} = \sin\theta r \partial\Omega/\partial r$ is the shear and $2rV_{\phi r} = \sin\theta\partial(r^2\Omega)/\partial r$ is the vorticity. The dots indicate buoyancy and meridional currents. The forms of the turbulent diffusivities entering the shear part D_s, vorticity part D_v, rigid rotation Ω_0 and differential rotation $\Omega \equiv \Omega(r, \theta)$ are also derived. Previous models have only the shear term. The vorticity term gives rise to a true diffusion-like equation for the angular momentum which now reads $\frac{\partial}{\partial t}(r^2\Omega) = r^{-2}\frac{\partial}{\partial r}\left(r^4 D_s \frac{\partial\Omega}{\partial r}\right) + r^{-2}\frac{\partial}{\partial r}\left[r^2 D_v \frac{\partial}{\partial r}(r^2\Omega)\right] + \ldots$.

(ii) Mean temperature equation. Differential rotation alters the mean temperature equation. In the stationary case, the new flux conservation law reads (χ is the radiative diffusivity) $\nabla + K_{\mathrm{h}}\chi^{-1}(\nabla - \nabla_{\mathrm{ad}}) + \nabla_\Omega = \nabla_r$, where the new term is given by $\nabla_\Omega = (H_{\mathrm{p}}/c_{\mathrm{p}}\chi T)R_{r\phi}\bar{u}_\phi$.

(iii) Tensorial diffusivities. The turbulent flux of a scalar ϕ (like T and μ) is shown to have the form $J_i^\phi = -D_{ij}^\phi \frac{\partial\Phi}{\partial x_j}$, where the D_{ij} are tensorial diffusivities. They are shown to be functions of the external source of energy (e.g. flux of gravity waves), rigid-body rotation, differential rotation, meridional currents, T–μ gradients and Peclet number *Pe* which characterizes the role of radiative losses.

(iv) Mixing and advection. The tensorial nature of the diffusivities D_{ij} has an immediate consequence: the symmetric part D_{ij}^{s} gives rise to mixing (by diffusion) while the anti-symmetric part D_{ij}^{a} gives rise to advection which cannot be represented by a diffusion coefficient. The equation describing a mean scalar field Φ is therefore $\frac{\partial\Phi}{\partial t} + (\bar{\boldsymbol{u}} + \boldsymbol{u}^*)\cdot\nabla\Phi = \frac{\partial}{\partial x_i}\left(D_{ij}^{\mathrm{s}}\frac{\partial\Phi}{\partial x_j}\right)$, $u_i^* = \frac{\partial}{\partial x_j}D_{ij}^{\mathrm{a}}$. Thus, even without a mean velocity field $\bar{\boldsymbol{u}}$, there is an advective term $\boldsymbol{u}^*$ arising from turbulence alone. The advective nature of turbulence was not accounted for in previous studies which have therefore underestimated the full potential of turbulent motion.

(v) Peclet number dependence. Radiative losses are an important part of the physical picture, for they weaken the temperature gradient, and thus reduce the effect of stable stratification and ultimately enhance mixing. The Peclet number dependence is accounted for in the model.

(vi) Shear-induced versus wave-induced mixing. In this formalism, the dichotomy between the two processes no longer exists, since we show that the flux of gravity waves, treated as an external source of energy, is a natural ingredient of the formalism.

Key words: turbulence – stars: rotation.

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1 THE PROBLEM

For many years, the convective zone of a star attracted most of the attention, and at present a variety of different models have become available, varying from an improved mixing length which accounts for the full eddy spectrum (Canuto & Mazzitelli 1991) to modern Reynolds stresses-based models that include the essential feature of non-locality.

By contrast, our understanding and ability to describe the dynamical properties of the stably stratified radiative zones are more rudimentary. Helioseismology has greatly helped to determine the acoustic structure of the Sun, but not the thermal and chemical structure which are due to mixing (by diffusion) and transport (by advection). We may cite three unsolved problems.

Data indicate that the solar Li abundance is some 200 times smaller than the initial abundance, an occurrence which has generally been viewed as an indication of 'deep but gentle mixing' several orders of magnitude smaller than the values in the convective zone but larger than the kinematic values. Such mixing would bring Li toward the interior of the Sun where it gets destroyed. The nature of the mixing is still debated, with rotational mixing and wave-induced mixing being the two major contenders (Montalban & Schatzman 1996; Pinsonneault 1997; Roxburgh 1997; Zahn 1997; Fritts, Vadas & Andreassen 1998).

A second unresolved problem is why the differential rotation that characterizes the outer 25 per cent radius of the Sun gives way to an almost rigid-body rotation in the remaining 75 per cent of the solar radius. Since no work can be extracted from rigid rotation, the most frequently discussed alternatives are shear and internal gravity waves. While the two processes are always discussed as being mutually exclusive, in our treatment we incorporate both of them into the same formalism.

The third problem is of a more conceptual nature. It is always assumed that turbulence of whatever origin only mixes but never advects, and yet it will be shown that for the case of scalars, turbulence gives rise to both mixing and advection, while for the case of angular momentum, only mixing occurs via the Reynolds stresses. Mixing and advection are quite different processes: mixing tends to smooth out density differences so as to lead to a homogeneous flow, while advection does somehow the opposite, since it brings together different parts of the flow which may have different densities.

Even an incomplete list of the work in this area brings out the flavour of the wide variety of different viewpoints as to which mechanism is more relevant. Since microscopic diffusion is insufficient, Charbonneau & Michaud (1998) added meridional currents, but Balachandran (1990) pointed out ensuing disagreement with observed data; a weak turbulence induced by rotation was added by Vauclair (1988), while gravity waves generated at the bottom of the convective zone were suggested by Garcia Lopez & Spruit (1991). While an arbitrary enhancement by a factor of 15 was needed to fit the data, we think that the factor is only ≈ 3 which stems from using a critical Richardson number $Ri(\mathrm{cr}) \approx 1$, as derived from non-linear stability analysis, rather than $Ri(\mathrm{cr}) = 1/4$. Richard et al. (1996) showed that meridional currents alone may explain Li^7 data provided that one tweaks adjustable parameters. Pinsonneault et al. (1989) suggested a model that explains the Li solar depletion, but the resulting $\Omega(r)$ in the interior of the Sun is far from flat as demanded by helioseismological data. Ventura et al. (1998) were able to reproduce the Li^7 abundance with an overshooting (OV) compatible with helioseismological data provided that there is a magnetic field. Schattl & Weiss (1999) have shown that an OV model based on a 2D simulation is not compatible with Li data. Talon & Charbonnel (1998) and Charbonnel & Talon (1999) assumed that the momentum diffusivity is identical to the concentration diffusivity which, in units of the radiative diffusivity, was taken to be a fixed fraction of the Richardson number. Schatzman, Zahn & Morel (2000) have concluded that shear instability is an unlikely source of mixing [their conclusion would have been just the opposite had they used $Ri(\mathrm{cr}) \approx 1$ instead of $Ri(\mathrm{cr}) = 1/4$]. Charbonneau et al. (1999) and Dikpati & Gilman (2001) have recently carried out a 2D linear stability analysis and concluded that, if there is an OV region with $|\nabla - \nabla_{\mathrm{ad}}| \approx 10^{-5}$, a shear instability sets in and by inference a shear-induced mixing. However, to translate that into a turbulent diffusivity of any kind cannot be achieved with a linear analysis, for it requires the inclusion of non-linearities, a considerably more demanding task.

The goal of this paper is not to favour one model over another. Rather, we work out the most complete model that present turbulence modelling allows us to construct for the turbulent fluxes of momentum, T and μ. This will help to boost the reliability of the model and the conclusions that will ensue once the model is applied to a star. In principle, to include waves, we would have to begin by splitting every field as follows:

$$\phi = \bar{\phi} + \phi'' + \tilde{\phi}.$$

Here, $\bar{\phi}$ is the mean field, ϕ'' the turbulent part and $\tilde{\phi}$ the part arising from the waves. As of today, the full Reynolds stress methodology has not yet been applied to this case. Here, we shall give the most complete treatment possible in the case in which we parametrize the effect of waves via the amount of energy that they contribute, a variable that will be left undetermined in our model. Admittedly, this will result in a dynamical model only for the first two components $\bar{\phi}$ and ϕ'', while the third component $\tilde{\phi}$ is not treated dynamically. Even so, we think that this work is a necessary step, for it improves and ties up several loose ends of previous treatments, not to mention its pedagogical value in preparation for a more complete model when all three terms in ϕ will be accounted for.

The model yields the Reynolds stresses R_{ij} and the tensor diffusivities D_{ij} as functions of the following variables:

$$\Pi, \Omega_0, \Omega(r, \theta), \bar{u}_r, \bar{u}_\theta, \nabla T, \nabla\mu, Pe.$$

Here, Π represents the energy source(s), Ω_0 and $\Omega(r, \theta)$ are the rigid and differential rotation so that the transition between the two regimes is accounted for without discontinuities (a requirement not met by present-day Reynolds stress models), $\bar{u}_r$ and $\bar{u}_\theta$ are the meridional currents, ∇T and $\nabla\mu$ are the T- and μ-gradients, and Pe is the Peclet number that governs the role of radiative losses which weaken the stable temperature gradient and favour mixing. The expressions for R_{ij} and D_{ij} are analytical.

2 GENERAL FORMULATION

2.1 Turbulent fluxes

The fields of velocity, temperature and mean molecular weight are split into an average part (denoted by an overbar) and a fluctuating part (denoted by a double prime). Using the notation $\mathrm{D}/\mathrm{D}t = \partial/\partial t + \bar{u}_i\partial/\partial x_i$, the dynamic equations for the mean velocity field $\bar{\boldsymbol{u}}$ and the mean scalar field Φ (e.g. T and μ) are of the form (Canuto

1997)

$$\bar{\rho}\frac{\mathrm{D}\bar{u}}{\mathrm{D}t}i = -\frac{\partial}{\partial x_j}(\bar{\rho}R_{ij}) + \ldots, \tag{1a}$$

$$\bar{\rho}\frac{\mathrm{D}\Phi}{\mathrm{D}t} = -\frac{\partial}{\partial x_i}(\bar{\rho}J_i^{\phi}) + \ldots, \tag{1b}$$

where the Reynolds stresses R_{ij} and the turbulent fluxes of the scalar field ϕ are defined as

$$\bar{\rho}R_{ij} = \overline{\rho u_i'' u_j''}, \tag{1c}$$

$$\bar{\rho}J_i^{\phi} = \overline{\rho u'' \phi''}. \tag{1d}$$

A turbulence model is used to express R_{ij} and J_i^{ϕ} in terms of the large-scale fields $\bar{\boldsymbol{u}}$ and Φ.

2.2 Scalar fields: mixing and advecting

It will be shown that J_i^{ϕ} is given by the following general expression:

$$J_i^{\phi} = -D_{ij}^{\phi}\frac{\partial\Phi}{\partial x_j}, \tag{2a}$$

where the D_{ij}^{ϕ} are the tensorial diffusivities. An immediate consequence of the tensorial nature of the diffusivities D_{ij} is the following. Let us split the diffusivity tensor into its symmetric (s) and antisymmetric (a) parts:

$$D_{ij} = \tfrac{1}{2}(D_{ij} + D_{ji}) + \tfrac{1}{2}(D_{ij} - D_{ji}) = D_{ij}^{\mathrm{s}} + D_{ij}^{\mathrm{a}}. \tag{2b}$$

Then the dynamic equation (1b) for ($\bar{\rho} = 1$) Φ becomes

$$\frac{\partial\Phi}{\partial t} + (\bar{\boldsymbol{u}} + \boldsymbol{u}^*)\cdot\nabla\Phi = \frac{\partial}{\partial x_i}\left(D_{ij}^{\mathrm{s}}\frac{\partial\Phi}{\partial x_j}\right), \tag{2c}$$

$$u_i^* = \frac{\partial}{\partial x_j}D_{ij}^{\mathrm{a}}. \tag{2d}$$

Only the symmetric part of the diffusivity tensor contributes to true diffusion (right-hand side of equation 2c), while the antisymmetric part gives rise to an advective velocity $\boldsymbol{u}^*$ (referred to as the bolus velocity). Thus *turbulence gives rise to both mixing by diffusion via the symmetric* D_{ij}^{s} *and stirring by advection via the antisymmetric* D_{ij}^{a}. This general argument shows that it is not correct to ascribe, as is often done, diffusion to turbulence and advection to a mean flow, since turbulence contributes to both.

To give a specific example, consider a 1D case. Since the density $\rho(T, \mu)$ is a function of both temperature and mean molecular weight, the Boussinesq approximation gives

$$\rho''/\rho = -\alpha_{\mathrm{h}}T'' + \alpha_{\mu}\mu'', \tag{3a}$$

where $\alpha_{\mathrm{h}} = -\rho^{-1}\partial\rho/\partial T$ and $\alpha_{\mu} = \rho^{-1}\partial\rho/\partial\mu$ are the expansion coefficients at constant pressure. Thus the buoyancy (or mass) flux obtained from (1d) with $\phi'' = \rho''$ is given in terms of the heat ($\phi'' = T''$) and $\mu(\phi'' = \mu'')$ fluxes as

$$J_i^{\rho} = -\alpha_{\mathrm{h}}J_i^{\mathrm{h}} + \alpha_{\mu}J_i^{\mu}. \tag{3b}$$

Using (2a), the heat, μ and mass fluxes are given by ($\rho = 1$ for simplicity)

$$J_{\mathrm{h}} = D_{\mathrm{h}}\left[-\frac{\partial T}{\partial z} + \left(\frac{\partial T}{\partial z}\right)_{\mathrm{ad}}\right], \quad J_{\mu} = -D_{\mu}\frac{\partial\mu}{\partial z},$$

$$J_{\rho} = -D_{\rho}\frac{\partial\rho}{\partial z}. \tag{3c}$$

Changing to the standard dimensionless gradients ∇, we obtain

$$D_{\rho} = (D_{\mathrm{h}} - R_{\mu}D_{\mu})(1 - R_{\mu})^{-1}, \tag{4a}$$

$$R_{\mu} = \nabla_{\mu}(\nabla - \nabla_{\mathrm{ad}})^{-1}, \tag{4b}$$

which shows that the mass diffusivity D_{ρ} is fully expressed in terms of $D_{\mathrm{h},\mu}$. It follows that since overshooting is a mass transport, it must be described by J_{ρ} rather than by J_{h}, as is usually done.

2.3 Mean *T*-equation

The complete form of equation (1b) for the mean temperature T is as follows (Canuto 1997):

$$\bar{\rho}\frac{\mathrm{D}}{\mathrm{D}t}(\bar{h} + E + E_u + G) = -\nabla\cdot(\boldsymbol{F}^{\mathrm{rad}} + \boldsymbol{F}^{\mathrm{h}} + \boldsymbol{F}^{\kappa\epsilon} + \tilde{\boldsymbol{\tau}}) + \frac{\partial P}{\partial t}. \tag{5a}$$

This is a true energy conservation law, since on the left-hand side it contains the sum of all the energies, thermal ($\bar{h} = c_{\mathrm{v}}T + \bar{p}/\bar{\rho}$ is the mean enthalpy which is equal to $c_{\mathrm{p}}T$ for a perfect gas), kinetic (turbulent kinetic energy $E = \frac{1}{2}\tau_{ii}$ and mean flow kinetic energy $E_u = \frac{1}{2}\bar{u}_i\bar{u}_i$) and gravitational ($\mathrm{D}G/\mathrm{D}t = g_i\bar{u}_i$). On the right-hand side of (5a) we have the divergences of the radiative flux $\boldsymbol{F}^{\mathrm{rad}}$, the thermal flux $\boldsymbol{F}^{\mathrm{h}}$, the flux of the turbulent kinetic energy $\boldsymbol{F}^{\kappa\epsilon}$ and the flux $\tilde{\tau}$ of the Reynolds stresses by the mean velocity field:

$$F_i^{\kappa\epsilon} = \tfrac{1}{2}\overline{\rho u_i'' u_k'' u_k''}, \quad \tilde{\tau}_i = \bar{\rho}R_{ij}\bar{u}_j. \tag{5b}$$

A model is needed not only for $\boldsymbol{F}^{\mathrm{h}}$ and R_{ij} (see below), but also for the third-order moment $F_i^{\kappa\epsilon}$ (Canuto, Cheng & Howard 2001; Kupka, in preparation). The *T*-equation usually employed, namely

$$\bar{\rho}\frac{\mathrm{D}\bar{h}}{\mathrm{D}t} = -\nabla\cdot(\boldsymbol{F}^{\mathrm{rad}} + \boldsymbol{F}^{\mathrm{h}}), \tag{5c}$$

is a considerable simplification of (5a). Particularly interesting is the new term $\tilde{\boldsymbol{\tau}}$ which physically corresponds to the transport of turbulence (i.e. of the Reynolds stresses) by the mean flow itself, a process that enhances mixing. To visualize the implications of this new term, consider the stationary limit $\partial/\partial t \equiv 0$ of (5a). We have

$$F_i^{\mathrm{rad}} + F_i^{\mathrm{h}} + F_i^{\kappa\epsilon} + \rho\bar{u}_j[(\bar{h} + E + E_u + G)\delta_{ij} + R_{ij}] = \text{constant}, \tag{5d}$$

which substitutes the standard flux conservation law:

$$F_i^{\mathrm{rad}} + F_i^{\mathrm{h}} = \text{constant}. \tag{5e}$$

Consider the r-component of (5d). Differential rotation will give rise to a new term:

$$F_r^{\mathrm{rad}} + F_r^{\mathrm{h}} + F_r^{\kappa\epsilon} + \rho\bar{u}_{\phi}R_{r\phi} = \text{constant} \tag{5f}$$

or, neglecting for a moment $F^{\kappa\epsilon}$,

$$\nabla + K_{\mathrm{h}}\chi^{-1}(\nabla - \nabla_{\mathrm{ad}}) + \nabla_{\Omega} = \nabla_r, \tag{5g}$$

$$\nabla_{\Omega} = (H_{\mathrm{p}}/c_{\mathrm{p}}T\chi)R_{r\phi}\bar{u}_{\phi}. \tag{5h}$$

The form of the Reynolds stresses R_{ij} is discussed below.

2.4 Velocity field

The dynamic equation (1a) for the mean velocity field $\bar{u}_i$ defined via the mass (Favre) average (Canuto 1997),

$$\bar{\rho}\bar{u}_i = \overline{\rho u_i}, \quad \overline{\rho u_i''} = 0, \tag{6a}$$

becomes

$$\frac{\partial}{\partial t}(\bar{\rho}\bar{u}_i) + \frac{\partial}{\partial x_j}(\bar{\rho}\bar{u}_i\bar{u}_j + \bar{\rho}R_{ij}) = -\bar{\rho}g_i - \frac{\partial P}{\partial x_i} - 2\bar{\rho}\epsilon_{ijk}\bar{u}_k\Omega_j^0, \tag{6b}$$

where Ω^0 represents rigid rotation and R_{ij} are the Reynolds stresses defined in (1c).

Finally, the mean density $\bar{\rho}$ satisfies the dynamic equation

$$\frac{\partial\bar{\rho}}{\partial t} + \frac{\partial}{\partial x_i}(\bar{\rho}\bar{u}_i) = 0. \tag{6c}$$

In spherical coordinates, equations (6b) become ($\Gamma \equiv \sin\theta$; $\rho \equiv \bar{\rho}$)

$$-r^3\frac{\partial}{\partial t}(\rho\bar{u}_\phi) = \frac{\partial}{\partial r}(r^3\psi_{r\phi}) + r^2\Gamma^{-2}\frac{\partial}{\partial\theta}(\Gamma^2\psi_{\theta\phi}) + 2\rho r^3\Omega_0(\Gamma\bar{u}_r + \cos\theta\bar{u}_\theta), \tag{6d}$$

$$\frac{\partial}{\partial t}(\rho\bar{u}_r) = A_1 - \frac{\partial P}{\partial r} + \rho\frac{\partial\Phi}{\partial r} + 2\rho\Omega_0\sin\theta\bar{u}_\phi, \tag{6e}$$

$$\frac{\partial}{\partial t}(\rho\bar{u}_\theta) = A_2 - r^{-1}\frac{\partial P}{\partial\theta} + \rho r^{-1}\frac{\partial\Phi}{\partial\theta} + 2\rho\Omega_0\cos\theta\bar{u}_\phi \tag{6f}$$

where, for the sake of brevity, we have defined the variables

$$A_1 \equiv -r^{-2}\frac{\partial}{\partial r}(r^2\psi_{rr}) - (r\Gamma)^{-1}\frac{\partial}{\partial\theta}(\Gamma\psi_{r\theta}) + r^{-1}(\psi_{\theta\theta} + \psi_{\phi\phi}), \tag{6g}$$

$$A_2 \equiv -r^{-3}\frac{\partial}{\partial r}(r^3\psi_{r\theta}) - (r\Gamma)^{-1}\frac{\partial}{\partial\theta}(\Gamma\psi_{\theta\theta}) + (rtg\theta)^{-1}\psi_{\phi\phi}, \tag{6h}$$

where

$$\psi_{ij} \equiv \bar{\rho}(T_{ij} + R_{ij}), \tag{7a}$$

$$T_{r\phi} = \bar{u}_r\bar{u}_\phi, \quad T_{\theta\phi} = \bar{u}_\theta\bar{u}_\phi, \tag{7b}$$

$$\bar{\rho}R_{r\phi} = \overline{\rho u_r''u_\phi''}, \qquad \bar{\rho}R_{\theta\phi} = \overline{\rho u_\theta''u_\phi''}. \tag{7c}$$

2.5 Angular momentum

The total angular momentum L of a parcel of unit mass is the sum of absolute (solid-body rotation Ω_0) contributions, and those relative to the rotating star $\Omega(r,\theta)$:

$$L = L_0 + L_r = r\Gamma(r\Gamma\Omega_0 + \bar{u}_\phi). \tag{8a}$$

It can easily be proved that L satisfies the following conservation law:

$$\frac{\partial}{\partial t}(\rho L) + \nabla\cdot(\rho\boldsymbol{F}_L) = 0. \tag{8b}$$

$\boldsymbol{F}_L$ is the flux of angular momentum, the (r,θ) components of which are given by

$$F_L^r = L\bar{u}_r + r\Gamma R_{r\phi}, \tag{8c}$$

$$F_L^\theta = L\bar{u}_\theta + r\Gamma R_{\theta\phi}. \tag{8d}$$

3 REYNOLDS STRESSES: PREVIOUS MODELS

If one uses the standard Reynolds stress model,

$$R_{r\phi} = -2D_\mathrm{m}S_{r\phi}. \tag{9a}$$

where D_m is a momentum diffusivity.

Equation (6d) yields the following dynamic equation for the angular momentum,

$$\frac{\partial}{\partial t}(r^2\Omega) = r^{-2}\frac{\partial}{\partial r}\left(r^4D_\mathrm{m}\frac{\partial\Omega}{\partial r}\right) + \ldots, \tag{9b}$$

that is most frequently used (Chaboyer & Zahn 1992, equation 18; Zahn 1992, equation 2.4; Pinsonneault et al. 1989, equation 3). There are several problems with (9a,b). First, equation (9b) is usually called a ‘diffusion equation’, but it is not so, since a true diffusion equation must have the form

$$\frac{\partial}{\partial t}(r^2\Omega) = r^{-2}\frac{\partial}{\partial r}\left[r^2D_\mathrm{m}\frac{\partial}{\partial r}(r^2\Omega)\right] + \ldots. \tag{9c}$$

Secondly, model (9a) is just

$$R_{ij} = -2D_\mathrm{m}S_{ij}, \tag{10a}$$

and since shear and vorticity,

$$2S_{ij} = \bar{u}_{i,j} + \bar{u}_{j,i}, \qquad 2V_{ij} = \bar{u}_{i,j} - \bar{u}_{j,i}, \tag{10b}$$

are two independent, orthogonal tensors, the question arises as to why (10a) contains shear and not vorticity. In general, one should expect an expression of the form

$$R_{ij} = R_{ij}(S_{ij}, V_{ij}). \tag{10c}$$

From the physical point of view, we recall that shear is contributed by the large scales while vorticity is contributed mostly by the smaller scales, and thus the presence of both shear and vorticity in (10c) is demanded by general considerations. In Section 5 we shall show that the $R_{\phi r}$-component of the Reynolds stresses has the form

$$R_{\phi r} = -2D_sS_{\phi r} - 2D_vV_{\phi r} - D_0\Omega_0 - D_1\Omega + \ldots, \tag{10d}$$

where $2S_{\phi r} = r\sin\theta\,\partial\Omega/\partial r$ is the shear and $2V_{\phi r} = r\sin\theta\partial\Omega/\partial r$ is the vorticity. The remaining terms represent buoyancy and meridional currents. Thus we see that the momentum flux is contributed by

$$R_{\phi r} \sim \text{rigid rot.} + \text{diff. rot.} + \text{grad}\ \Omega + \text{grad}\ L. \tag{10e}$$

Over the years, several attempts have been made to improve (10a). Durney & Spruit (1979) and Durney (2000) used a mixing-length model to construct the following model:

$$R_{r\phi} = -\sin\theta\left(A\Omega + Br\frac{\partial\Omega}{\partial r}\right), \tag{11a}$$

while Rudiger (1989) and Kuker, Rudiger & Kichatinov (1993) suggested the relation

$$R_{r\phi} = \Lambda_\mathrm{v}\sin\theta\Omega – \upsilon_\mathrm{vh}\cos^2\theta\frac{\partial\Omega}{\partial\theta} - \upsilon_\mathrm{vv}\sin\theta r\frac{\partial\Omega}{\partial r}, \tag{11b}$$

where the subscripts v and h stand for vertical and horizontal, respectively. These models do not yield the same expressions for $R_{r\phi}$ and they are also unable to evaluate the turbulent viscosities A, B, ν and Λ.

(i) *R_{ij} should depend on buoyancy.* Since the Navier–Stokes equations for the fluctuating velocity u_i'' contain the term

$$\frac{\partial}{\partial t}u_i'' = \ldots + g_i\rho'', \tag{11c}$$

it is clear that in constructing the Reynolds stresses R_{ij} there will necessarily be a buoyancy term of the form

$$B_{ij} \approx g_i \overline{u_j'' \rho''} + g_j \overline{u_i' \rho''} \approx B_{ij}(\nabla T, \nabla \mu), \tag{11d}$$

where ∇T and $\nabla \mu$ are the T- and μ-gradients. Buoyancy terms are not accounted for in (10a).

(ii) *R_{ij} should depend on the meridional currents.* Since equation (10c) shows that R_{ij} must depend on both shear and vorticity, it will naturally also depend on the meridional currents $\bar{u}_r$ and $\bar{u}_\theta$:

$$S_{rr} = \bar{u}_{r,r}, \qquad rS_{\theta\theta} = \bar{u}_r + \bar{u}_{\theta,\theta},$$

$$-2rV_{r\phi} = \sin\theta \frac{\partial}{\partial r}(r^2\Omega), \qquad rS_{\phi\phi} = \bar{u}_r + \bar{u}_\theta tg^{-1}\theta,$$

$$-2V_{\theta\phi} = 2\Omega\cos\theta + \sin\theta\frac{\partial\Omega}{\partial\theta},$$

$$2rS_{r\theta} = \bar{u}_{r,\theta} - \bar{u}_\theta + r\bar{u}_{\theta,r}, \qquad 2rV_{r\theta} = \bar{u}_{r,\theta} - \bar{u}_\theta - r\bar{u}_{\theta,r}. \tag{11e}$$

This means that the meridional currents will enter the angular momentum equation (8b),

$$\frac{\partial}{\partial t}(\rho L) + r^{-2}\frac{\partial}{\partial r}(\rho r^2 L\bar{u}_r) + r^{-2}\frac{\partial}{\partial r}[\rho r\Gamma R_{r\phi}(\bar{u}_r, \bar{u}_\theta)] + \ldots = 0, \tag{11f}$$

not only via the advective second term but also via the 'turbulent' $R_{r\phi}$ which now also depends on the meridional currents, as we have explicitly indicated in (11f). In previous models (Zahn 1997), the meridional currents only appear in the second term in (11f).

In summary, the standard Reynolds stress model (9a) does not provide a complete expression that includes rigid rotation, differential rotation, meridional currents and buoyancy terms, a functional dependence that we express as

$$R_{ij}[\Pi,\ \Omega_0,\ \Omega(r,\ \theta),\ \bar{u}_r,\ \bar{u}_\theta,\ \nabla T,\ \nabla\mu,\ Pe]. \tag{11g}$$

The same dependence also holds true for all the diffusivity tensors D_{ij}. In the next sections we shall derive (11g).

4 REYNOLDS STRESSES: DERIVATION FROM THE NAVIER–STOKES EQUATIONS

Using the Navier–Stokes equations for the turbulent fields, the traceless Reynolds stresses (E is the turbulent kinetic energy)

$$b_{ij} = R_{ij} - \tfrac{2}{3}E\delta_{ij}, \qquad E = \tfrac{1}{2}R_{ii} \tag{12}$$

satisfy the following dynamic equation (Canuto 1999):

$$\frac{\mathrm{D}}{\mathrm{D}t}b_{ij} + \mathrm{D}_f(b) = -\frac{4}{3}KS_{ij} - \Sigma_{ij} - Z^*_{ij} - B^\rho_{ij} - \Pi_{ij}, \tag{13a}$$

where D_f represents the diffusion of b_{ij}. The pressure correlation tensor Π_{ij} and the other tensors are defined as ($a_i \equiv \partial a/\partial x_i$)

$$\Pi_{ij} = \overline{u_i p_{,j}} + \overline{u_j p_{,i}} - \tfrac{1}{3}\delta_{ij}\overline{u_k p_{,k}}, \tag{13b}$$

$$\Sigma_{ij} = b_{ik}S_{jk} + b_{jk}S_{ik} - \tfrac{2}{3}\delta_{ij}b_{\mathrm{km}}S_{\mathrm{km}}, \tag{13c}$$

$$Z^*_{ij} = b_{ik}V^*_{jk} + b_{jk}V^*_{ik}, \tag{13d}$$

$$V^*_{ij} = V_{ij} - 2\epsilon_{ijp}\Omega_{0p}, \tag{13e}$$

$$B^\rho_{ij} = g(\lambda_i J^\rho_j + \lambda_j J^\rho_i) - \tfrac{2}{3}g\delta_{ij}\lambda_k J^\rho_k, \tag{13f}$$

where shear S_{ij}, vorticity V_{ij} and buoyancy B_{ij} have already been defined, and where

$$\lambda_i = -(\bar{\rho}g)^{-1}\frac{\partial P}{\partial x_i}. \tag{13g}$$

4.1 Pressure correlations

The pressure correlations represented by Π_{ij} are the most difficult part to model. A rather long discussion is presented by Canuto (1994). It was shown that the Π_{ij} contain slow, rapid and buoyancy contributions, and that the specific form is given by (Canuto 1999)

$$\Pi_{ij} = \Pi_{ij}(\mathrm{slow}) - \tfrac{4}{5}KS_{ij} - \alpha_1\Sigma_{ij} - \alpha_2\tilde{Z}_{ij} + (1-\beta_5)B_{ij}, \tag{14a}$$

where $\alpha_{1,2}$ and β_5 are coefficients to be given later. In addition,

$$\tilde{Z}_{ij} = b_{ik}\tilde{V}_{jk} + b_{jk}\tilde{V}_{ik}, \tag{14b}$$

$$\tilde{V}_{ij} = V_{ij} - \epsilon_{ijp}\Omega_{0p}. \tag{14c}$$

Notice that in the pressure correlations the vorticity $\tilde{V}_{ij}$ does not have the factor of 2 that usually appears in the Coriolis term. The slow part is given by Rotta's 'return to isotropy' model:

$$\Pi_{ij}(\mathrm{slow}) = A\tau^{-1}b_{ij}, \tag{14d}$$

where A is a numerical coefficient,

$$\tau = 2E\epsilon^{-1} \tag{14e}$$

is the eddy turnover time, and ϵ is the rate of dissipation of E (Section 9). Thus equation (13a) becomes

$$\frac{\mathrm{D}}{\mathrm{D}t}b_{ij} + \mathrm{D_f}(b) = -A\tau^{-1}b_{ij} - \frac{8}{15}ES_{ij} - (1-\alpha_1)\Sigma_{ij} - (1-\alpha_2)Z_{ij} - \beta_5 B^\rho_{ij}, \tag{14f}$$

where

$$Z_{ij} = b_{ik}V_{jk} + b_{jk}V_{ik} - p(b_{ik}\epsilon_{jkp} + b_{jk}\epsilon_{ikp})\Omega_{0p}, \tag{14g}$$

$$p \equiv (2-\alpha_2)(1-\alpha_2)^{-1}. \tag{14h}$$

Physically, we can interpret the various terms on the right-hand side of (14f) as the return to isotropy, shear, the non-linear return to isotropy, vorticity + rotation and buoyancy.

The algebraic Reynolds stress model corresponds to neglecting the left-hand side of (14f) so that the equation for b_{ij} becomes a set of algebraic equations:

$$A\tau^{-1}b_{ij} = -\frac{8}{15}ES_{ij} - (1-\alpha_1)\Sigma_{ij} - (1-\alpha_2)Z_{ij} - \beta_5 B^\rho_{ij}. \tag{15a}$$

Let us note that model (10a) corresponds to keeping only the first term in (15a). The suggested values of the constants are $A = 5$, $\alpha_1 = 0.98$, $\alpha_2 = 0.568$ and $\beta_5 = 1/2$.

5 REYNOLDS STRESSES: EXPLICIT FORM

Here we present the explicit form of the Reynolds stresses as from equation (15a). For ease of notation, we employ the dimensionless variables

$$r_{ij} \equiv R_{ij}/E, \qquad \tilde{\Omega}_0 \equiv \tau[\Omega(r,\theta) + p\Omega_0], \tag{15b}$$

$$\Omega_1 \equiv \tau\sin\theta\frac{\partial\Omega}{\partial\theta}, \quad \Omega_2 \equiv \tau r\sin\theta\frac{\partial\Omega}{\partial r}. \tag{15c}$$

The explicit forms of the R_{ij} are as follows.

5.1 $R_{r\phi}$

We have

$$q_0 r_{r\phi} = g_0\tilde{\Omega}_0 + g_1\Omega_1 + g_2\Omega_2 + 2\beta_5\epsilon^{-1}B_{r\phi} + g_{\mathrm{m}}, \tag{16}$$

where

$$q_0 = A + (1-\alpha_1)\tau r^{-1}(\bar{u}_r + r\bar{u}_{r,r} + \bar{u}_\theta tg^{-1}\theta),$$

$$g_0 = (\alpha_2 - 1)(r_{rr}\sin\theta - r_{\phi\phi}\sin\theta + r_{r\theta}\cos\theta),$$

$$2g_1 = (\alpha_1 + \alpha_2 - 2)r_{r\theta},$$

$$2g_2 = 4(1/5 - \alpha_1/3) + (\alpha_1 + \alpha_2 - 2)r_{rr} + (\alpha_1 - \alpha_2)r_{\phi\phi},$$

$$2g_{\mathrm{m}} = r_{\theta\phi}[(\alpha_1 + \alpha_2 - 2)\tau r^{-1}(\bar{u}_{r,\theta} - \bar{u}_\theta) + (\alpha_1 - \alpha_2)\tau\bar{u}_{\theta,r}]. \tag{17}$$

Several comments are in order.

(1) The first three terms in (16) can be combined to yield

$$R_{r\phi} = -2D_s S_{\phi r} - 2D_v V_{\phi r} - D_0\Omega_0 - D_1\Omega + \ldots, \tag{18a}$$

where the diffusivities $D_{s,v,0,1}$ corresponding to shear, vorticity, rigid rotation and differential rotation are given by

$$D_s = \frac{4}{15}q_0^{-1}\tau E,$$

$$2D_v = q_0^{-1}(1-\alpha_2)\tau(R_{rr} - R_{\phi\phi}),$$

$$D_1 = q_0^{-1}(1-\alpha_2)\tau R_{r\theta}\cos\theta,$$

$$D_0 = q_0^{-1}(2-\alpha_2)\tau[(R_{rr} - R_{\phi\phi})\sin\theta + R_{r\theta}\cos\theta]. \tag{18b}$$

In deriving (18b), we used the fact that $\alpha_1 \approx 1$. Thus the momentum flux $R_{r\phi}$ is contributed by the following processes:

$$R_{r\phi} \sim \Omega_0 + \Omega(r,\theta) + \text{ grad } \Omega + \text{ grad } L. \tag{18c}$$

The standard model (9a) has only the first term in (18a). It is important to stress that D_s and $D_{v,1,0}$ depend on two different processes: the first depends on $\alpha_1 \approx 1$ while the others depend on α_2. As one observes from the general structure of the Reynolds stresses (15a), the coefficient α_1 enters the equation via the shear terms Σ_{ij} as equation (13c) shows. On the other hand, the $D_{v,1,0}$ are entirely due to the $(1-\alpha_2)$ coefficient in (15a) which appears only via the presence of vorticity Z_{ij} defined in equation (14g).

(2) Once substituted into the angular momentum equations (8a,b), the vorticity term of (18a) gives rise to a true diffusion term for the angular momentum:

$$\frac{\partial}{\partial t}(r^2\Omega) = r^{-2}\frac{\partial}{\partial r}\left[r^2 D_v\frac{\partial}{\partial r}(r^2\Omega)\right] + r^{-2}\frac{\partial}{\partial r}\left(r^4 D_s\frac{\partial\Omega}{\partial r}\right) + \ldots. \tag{18d}$$

(3) Without the meridional currents and without buoyancy, $R_{r\phi}$ resembles models (11a,b) but the similarity is only superficial. In fact, a key feature is that heuristic models like (11a,b) cannot capture the fact that the functions g are themselves functions of the other stresses.

(4) The expression for $R_{r\phi}$ contains g_{m} which connects $R_{r\phi}$ to $R_{\theta\phi}$ via the meridional currents.

(5) Meridional currents contribute not only to g_{m} but also to q_0.

(6) We have separated the contribution arising from buoyancy, the term $B_{r\phi}$ which will be computed later.

(7) the DS model (Durney 2000, equation 19) has no meridional contributions, and g_0 has the first two terms but not the $r_{r\theta}$-term or the buoyancy term $B_{r\phi}$.

5.2 $R_{\theta\phi}$

$$q_1 r_{\theta\phi} = f_0\tilde{\Omega}_0 + f_1\Omega_1 + f_2\Omega_2 + 2\beta_5\epsilon^{-1}B_{\theta\phi} + f_{\mathrm{m}}. \tag{19}$$

Here, we have

$$q_1 = A + (1-\alpha_1)r^{-1}\tau(2\bar{u}_r + \bar{u}_{\theta,\theta} + \bar{u}_\theta tg^{-1}\theta),$$

$$f_0 = (\alpha_2 - 1)(r_{r\theta}\sin\theta + r_{\theta\theta}\cos\theta - r_{\phi\phi}\cos\theta),$$

$$2f_1 = 4(1/5 - \alpha_1/3) + (\alpha_1 + \alpha_2 - 2)r_{\theta\theta} + (\alpha_1 - \alpha_2)r_{\phi\phi},$$

$$2f_2 = (\alpha_1 + \alpha_2 - 2)r_{r\theta}, \tag{20}$$

$$2f_{\mathrm{m}} = r_{r\phi}[(\alpha_1 + \alpha_2 - 2)\tau\bar{u}_{\theta,r} + (\alpha_1 - \alpha_2)\tau r^{-1}(\bar{u}_{r,\theta} - \bar{u}_\theta)]. \tag{21}$$

5.3 $R_{r\theta}$

$$q_2 r_{r\theta} = h_0\tilde{\Omega}_0 + h_1\Omega_1 + h_2\Omega_2 + 2\beta_5\epsilon^{-1}B_{r\theta} + h_{\mathrm{m}}, \tag{22}$$

where

$$q_2 = A + (1-\alpha_1)r^{-1}\tau(\bar{u}_r + \bar{u}_{\theta,\theta} + r\bar{u}_{r,r}),$$

$$h_0 = (\alpha_2 - 1)(r_{r\phi}\cos\theta + r_{\theta\phi}\sin\theta),$$

$$2h_1 = (\alpha_1 - \alpha_2)r_{r\phi},$$

$$2h_2 = (\alpha_1 - \alpha_2)r_{\theta\phi}, \tag{23}$$

$$\begin{aligned} 2h_{\mathrm{m}} = {} & 4(1/5 - \alpha_1/3)\tau(\bar{u}_{\theta,r} + r^{-1}\bar{u}_{r,\theta} - r^{-1}\bar{u}_\theta) \\ & + r_{rr}[(\alpha_1 - \alpha_2)r^{-1}\tau(\bar{u}_{r,\theta} - \bar{u}_\theta) + (\alpha_1 + \alpha_2 - 2)\tau\bar{u}_{\theta,r}] \\ & + r_{\theta\theta}[(\alpha_1 - \alpha_2)\tau\bar{u}_{\theta,r} + (\alpha_1 + \alpha_2 - 2)r^{-1}\tau(\bar{u}_{r,\theta} - \bar{u}_\theta)]. \end{aligned} \tag{24}$$

5.4 R_{rr}

$$q_3 r_{rr} = s_0\tilde{\Omega}_0 + s_1\Omega_1 + s_2\Omega_2 + 2\beta_5\epsilon^{-1}B_{rr} + s_{\mathrm{m}}, \tag{25}$$

where

$$q_3 = A + \tfrac{4}{3}(1-\alpha_1)\tau\bar{u}_{r,r},$$

$$s_0 = 2(1-\alpha_2)r_{r\phi}\sin\theta,$$

$$3s_1 = 2(1-\alpha_1)r_{\theta\phi},$$

$$3s_2 = (2 + \alpha_1 - 3\alpha_2)r_{r\phi}, \tag{26}$$

$$\begin{aligned} 3s_{\mathrm{m}} = {} & 2A + \tfrac{8}{15}(2 - 5\alpha_1)\tau\bar{u}_{r,r} \\ & + 4(\alpha_1 - 1)\tau r^{-1}(2\bar{u}_r + \bar{u}_{\theta,\theta} + \bar{u}_\theta tg^{-1}\theta) \\ & + r_{r\theta}[(\alpha_1 + 3\alpha_2 - 4)\tau r^{-1}(\bar{u}_{r,\theta} - \bar{u}_\theta) \\ & + (2 + \alpha_1 - 3\alpha_2)\tau\bar{u}_{\theta,r}] - 2(\alpha_1 - 1)\tau r^{-1}[r_{\theta\theta}(\bar{u}_r + \bar{u}_{\theta,\theta}) \\ & + r_{\phi\phi}\tau(\bar{u}_r + \bar{u}_\theta tg^{-1}\theta)]. \end{aligned} \tag{27}$$

5.5 $R_{\phi\phi}$

$$q_4 r_{\phi\phi} = w_0\tilde{\Omega}_0 + w_1\Omega_1 + w_2\Omega_2 + 2\beta_5\epsilon^{-1}B_{\phi\phi} + w_{\mathrm{m}}, \tag{28}$$

where

$$q_4 = A + \tfrac{4}{3}(1-\alpha_1)\tau r^{-1}(\bar{u}_r + \bar{u}_\theta tg^{-1}\theta),$$

$$w_0 = 2(\alpha_2 - 1)(r_{r\phi}\sin\theta + r_{\theta\phi}\cos\theta),$$

$$3w_1 = (\alpha_1 + 3\alpha_2 - 4)r_{\theta\phi},$$

$$3w_2 = (\alpha_1 + 3\alpha_2 - 4)r_{r\phi}, \tag{29}$$

$$\begin{aligned}3w_{\mathrm{m}} = 2A &+ \tfrac{4}{3}(\alpha_1 - 1)\tau\bar{u}_{r,r} - \tfrac{4}{3}\mathrm{r}^{-1}\tau[(\alpha_1 + 1/5)\bar{u}_r \\ &+ (1-\alpha_1)\bar{u}_{\theta,\theta} + 2(\alpha_1 - 2/5)\bar{u}_\theta tg^{-1}\theta)] \\ &+ 2r_{r\theta}(1-\alpha_1)r^{-1}\tau(r\bar{u}_{\theta,r} + \bar{u}_{r,\theta} - \bar{u}_\theta) \\ &+ 2(1-\alpha_1)r_{rr}\tau\bar{u}_{r,r} + 2(1-\alpha_1)r^{-1}r_{\theta\theta}\tau(\bar{u}_r + \bar{u}_{\theta,\theta}).\end{aligned} \tag{30}$$

5.6 $R_{\theta\theta}$

$$r_{\theta\theta} = 2 - r_{rr} - r_{\phi\phi}. \tag{31}$$

The turbulent kinetic energy is given by

$$E = \tfrac{1}{2}(R_{rr} + R_{\theta\theta} + R_{\phi\phi}). \tag{32}$$

6 BUOYANCY FLUXES

As discussed earlier, the traceless buoyancy flux tensor B^ρ_{ij} is given by equation (13f), where the density flux J^ρ_i is defined in equation (3b). If we only keep $\partial P/\partial r$ and $\partial P/\partial\phi$, we have from (13g) the following relations:

$$-g\rho\lambda_r = \frac{\partial P}{\partial r}, \qquad -g\rho\lambda_\theta = r^{-1}\frac{\partial P}{\partial\theta}. \tag{33a}$$

Since, to first order, equations (6e, f) give

$$\frac{\partial P}{\partial r} = 2\rho\Omega_0\bar{u}_\phi\sin\theta - \rho g, \qquad r^{-1}\frac{\partial P}{\partial\theta} = 2\rho\Omega_0\bar{u}_\phi\cos\theta, \tag{33b}$$

we further have

$$\lambda_r = 1 - 2g^{-1}\Omega_0\sin\theta\bar{u}_\phi, \tag{33c}$$

$$\lambda_\theta = -2g^{-1}\Omega_0\cos\theta\bar{u}_\phi. \tag{33d}$$

If we assume that the only non-vanishing pressure gradients are in the r, θ-directions, the required buoyancy fluxes entering the expressions for R_{ij} are given by

$$\begin{aligned}&B_{r\phi} = g\lambda_r J^\rho_\phi, \qquad B_{\theta\phi} = g\lambda_\theta J^\rho_\phi, \qquad B_{r\theta} = g(\lambda_r J^\rho_\theta + \lambda_\theta J^\rho_r), \\ &B_{rr} = \tfrac{2}{3}g(2\lambda_r J^\rho_r - \lambda_\theta J^\rho_\theta), \qquad B_{\phi\phi} = -\tfrac{2}{3}g(\lambda_\theta J^\rho_\theta + \lambda_r J^\rho_r).\end{aligned} \tag{34}$$

7 HEAT AND μ-DIFFUSIVITIES

Since each J^ρ_i in (34) is given by equation (3b) in terms of the heat and μ-fluxes, we shall now express J^{h}_i and J^μ_i. Using previous results (Canuto 1999), we have

$$\text{heat flux}: \quad F^{\mathrm{h}}_i = c_p\bar{\rho}J^{\mathrm{h}}_i, \quad \bar{\rho}J^{\mathrm{h}}_i = \overline{\rho u''_i T''}, \tag{35a}$$

where

$$(\delta_{ij} + a_{ij})J^{\mathrm{h}}_j = \pi_1\tau R_{ij}\beta_j - \pi_1\pi_2 g\tau^2\alpha_\mu\lambda_i\beta_j J^\mu_j, \tag{35b}$$

$$a_{ij} = \pi_1\tau(S_{ij} + V_{ij}) - \pi_1 g\tau^2\lambda_i(\pi_3\alpha_{\mathrm{h}}\beta_j + \pi_2\alpha_\mu\mu_j), \tag{35c}$$

where

$$\beta_i = -\frac{\partial T}{\partial x_i} + \left(\frac{\partial T}{\partial x_i}\right)_{\mathrm{ad}}. \tag{35d}$$

We also have

$$\mu\text{-flux}: \quad F^\mu_i = \bar{\rho}J^\mu_i = \overline{\rho u''_i\mu''}, \tag{36a}$$

$$(\delta_{ij} + b_{ij})J^\mu_j = -\pi_1\tau R_{ij}\mu_j - \pi_1\pi_2 g\tau^2\alpha_{\mathrm{h}}\lambda_i\mu_j J^{\mathrm{h}}_j, \tag{36b}$$

$$b_{ij} = \pi_1\tau(S_{ij} + V_{ij}) - \pi_1 g\tau^2\lambda_i(\pi_2\alpha_{\mathrm{h}}\beta_j + \pi_3\alpha_\mu\mu_j), \tag{36c}$$

where

$$\mu_i = \frac{\partial\mu}{\partial x_i}. \tag{36d}$$

Equations (35)-(36) are a set of coupled, linear algebraic equations that can be solved to yield the fluxes $J^{\mathrm{h},\mu}_i$. The dimensionless time-scales $\pi(Pe)$ are discussed below.

With the above expressions for the heat and μ-fluxes, one then constructs the mass flux:

$$\text{mass flux} \ \ \bar{\rho}J^\rho_i = \overline{\rho u''_i\rho''}, \tag{37a}$$

$$J^\rho_i = -\alpha_{\mathrm{h}}J^{\mathrm{h}}_i + \alpha_\mu J^\mu_i. \tag{37b}$$

Using the Hamilton–Cayley theorem, equations (35b) and (36b) can be rewritten so as to exhibit the more familiar form (Canuto 1999)

$$J^{\mathrm{h}}_{\mathrm{i}} = D^{\mathrm{h}}_{ij}\beta_i, \ \ \text{etc.} \tag{37c}$$

However, since the tensors D^{h}_{ij} and D^μ_{ij} are rather complex, in practical applications equations (35)-(36) are preferable for they are a system of linear algebraic equations.

As an example, consider the case of zero μ-gradient and no shear/vorticity. Equation (35b) reduces to

$$(\delta_{ij} + a_{ij})J^{\mathrm{h}}_j = \pi_1\tau R_{ij}\beta_j, \tag{37d}$$

$$a_{ij} = -\pi_1\pi_3 g\tau^2\lambda_i\alpha_{\mathrm{h}}\beta_j. \tag{37e}$$

In a 1D case, we further have ($J^{\mathrm{h}}_3 \equiv J^{\mathrm{h}}$, $\beta_3 \equiv \beta$, $R_{ij} = R_{33} = \bar{w}^2$):

$$J_{\mathrm{h}} = D_{\mathrm{h}}\beta, \qquad D_{\mathrm{h}} = \pi_1\tau\bar{w}^2(1 + \pi_1\pi_3 N^2\tau^2)^{-1}, \tag{37f}$$

where D_{h} is the heat diffusivity and N^2 is the Brunt–Vaisala frequency:

$$N^2 = -g\alpha_{\mathrm{h}}\beta, \tag{37g}$$

which is positive for a stably stratified flow and negative in the convective case. Equation (37f) is the standard expression for the 1D heat flux.

8 PECLET NUMBER DEPENDENCE

The Peclet number dependence of the dimensionless time-scale π was derived by Canuto & Dubovikov (1998) to be

$$\pi_1 \equiv (4\pi^2)^{-1}Pe[1 + 5(4\pi^2)^{-1}(1 + \sigma_{\mathrm{t}}^{-1})Pe]^{-1}, \tag{38a}$$

$$\pi_3 \equiv 4(7\pi^2)^{-1}Pe[1 + 4(7\pi^2)^{-1}\sigma_{\mathrm{t}}^{-1}Pe]^{-1}, \tag{38b}$$

$$\pi_2 = \tfrac{1}{2}\pi_3, \tag{38c}$$

with $\sigma_{\mathrm{t}} = 0.72$. The Peclet number is defined as

$$Pe = \frac{4\pi^2}{125}\frac{E^2}{\epsilon\chi}. \tag{38d}$$

9 DYNAMIC EQUATIONS: GRAVITY WAVES

The above expressions for the fluxes, stresses and Peclet number involve two turbulence variables E and ϵ:

$$\tau = 2E/\epsilon, \tag{39a}$$

and thus one needs two additional equations for E and ϵ. The equation for E is

$$\frac{\partial E}{\partial t} + \nabla \cdot \boldsymbol{F}^{\kappa\epsilon} = P_{\mathrm{s}} + P_{\mathrm{b}} + \Pi - \epsilon, \tag{39b}$$

$$P_{\mathrm{s}} = -R_{ij}S_{ij}, \quad P_{\mathrm{b}} = -g\lambda_i J_i^{\rho}, \tag{39c}$$

where the flux of turbulent kinetic energy $\boldsymbol{F}^{\kappa\epsilon}$ represents the non-local character of turbulence (Canuto et al. 2001). Here, $\prod$ represents the possible contribution of sources other than shear, for example gravity waves, for which one can use the estimate by Kumar, Talon & Zahn (1999):

$$\Pi(\text{gravity waves}) \approx 0.1\,\text{per cent} \times (L/M)_{\text{sun}}$$
$$= 2 \times 10^{-3}\,\mathrm{cm^2\,s^{-3}} \tag{39d}$$

In the local limit, equation (39b) becomes

$$P_{\mathrm{S}} + P_{\mathrm{b}} + \Pi = \epsilon, \tag{39e}$$

which means that

$$\text{production} = \text{dissipation}. \tag{39f}$$

The equation for ϵ has been discussed previously (e.g. Canuto & Dubovikov 1998), and it could be utilized here too. However, we suggest a simpler model. To proceed, we begin with the Kolmogorov-like relation

$$\epsilon = E^{3/2}\Lambda^{-1}, \tag{40a}$$

where Λ is a mixing length to be discussed in what follows. Next, we introduce the dimensionless function

$$\psi \equiv (\tau N)^2, \tag{40b}$$

which measures the eddy turnover time $\tau = 2E/\epsilon$ in units of the time-scale of stratification N. Thus we can express both E and ϵ in terms of Λ and ψ as follows:

$$E = 4(\Lambda N)^2\psi^{-1}, \quad \epsilon = 8\Lambda^2 N^3 \psi^{-3/2}. \tag{40c}$$

Substituting (40c) in all the expressions for the Reynolds stresses R_{ij}, buoyancy fluxes J_i^{ρ} and Pe, one is left with only one unknown, ψ, which is determined by solving the algebraic equation (39e). The mixing length-scale Λ can be parametrized in two ways:

$$\Lambda = \alpha H_{\mathrm{p}}, \tag{41a}$$

which is the standard, although approximate, model in the absence of stable stratification; and

$$\Lambda = \alpha H_{\mathrm{p}} F(\psi), \tag{41b}$$

where $F(\psi)$ is a dimensionless function of ψ itself. The presence of $F(\psi)$ is the recognition that, in the presence of stable stratification, all length-scales are reduced. Deardorff (1980) first suggested the heuristic expression

$$\Lambda/l \approx wN^{-1}, \tag{41c}$$

where $w \approx E^{1/2}$ and l is the length-scale in the absence of stratification which one may parameterize using (41a). More recently, Cheng & Canuto (1994) have derived a more general model for Λ that encompasses (41c) as a limiting case. Using that result, we suggest that the ratio $\Lambda/l \equiv x$ be taken as the solution of the following cubic equation (see Cheng & Canuto 1994, equations 38a,b):

$$y^3 - y^2 + A(\psi)y + B(\psi) = 0, \tag{41d}$$

where

$$y \equiv x^{2/3}, \qquad A(\psi) = b\phi^{4/3}, \qquad B(\psi) = a\phi^2 - b\phi^{4/3}, \tag{41e}$$

$$\phi \equiv \tfrac{1}{2} c_\epsilon \psi^{1/2}, \qquad c_\epsilon \equiv \pi\left(\tfrac{2}{3K_0}\right)^{3/2}, \tag{41f}$$

$$3\pi^2 a = 2(S-1), \qquad a = 0.12\left(S - 1 + \tfrac{3}{2}S^{-1}\right)^{4/9}, \tag{41g}$$

$$S = 1 + \tfrac{1}{4}3^{1/2}Ko^{3/2}(R_{\mathrm{f}}^{-1} - 1). \tag{41h}$$

Ko is the Kolmogorov constant ($1.5 < Ko < 1.8$) and R_{f} is the flux Richardson number

$$R_{\mathrm{f}} = \frac{|P_{\mathrm{b}}|}{P_{\mathrm{S}} + \Pi} < 1. \tag{41i}$$

Alternatively, one can assume values of R_{f} and then check whether (41i) is satisfied. The ratio Λ/l is expected to be a decreasing function of ψ since the stronger the stratification N, the larger the reduction of the length-scale Λ. For example, for $\tau N = 40$ and $R_{\mathrm{f}} = 0.8$, equation (41b) gives $\Lambda/l = 0.2$.

10 PRACTICAL USE OF THE MODEL

As already discussed, in practical calculations, the most convenient form for the Reynolds stresses R_{ij} is given by the compact form (15a) which amounts to a system of linear algebraic equations for the various components of R_{ij}. The explicit form presented in Section 5 is convenient for discussing the physical content of the various components, but not for practical applications. The heat and μ fluxes needed to compute the buoyancy/mass flux tensor B_{ij} that is present in (15a) are computed using the definition (3b). The latter involves the heat and μ fluxes which are obtained by solving the set of linear algebraic equations (35)–(36). The two variables E and ϵ (we recall that $\tau = 2E/\epsilon$) must be written as in (40c) where ψ is the solution of the production = dissipation relation (39e). Finally, for Λ we have suggested two models, as shown by equations (41a,b).

11 CONCLUSIONS

We have presented the most updated Reynolds stresses R_{ij}, heat J_i^{h} and μ fluxes J_i^{μ} that today's turbulence models allow us to compute. This work is a considerable generalization not only of mixing-length-type models (Durney & Spruit 1979) but also of our own work (Canuto, Minotti & Schilling 1994). The Reynolds stresses R_{ij} depend explicitly on the buoyancy fluxes B_{ij} which appear in all the final expressions for R_{ij}. The heat and μ fluxes themselves depend on the large-scale flow $\bar{\boldsymbol{u}}(\bar{u}_r, \bar{u}_\phi, \bar{u}_\theta)$ as seen from equations (35)-(36). The present formalism can be applied to a variety of problems, of which we sample the most frequently discussed.

11.1 Transport of angular momentum

This has been studied by Pinsonneault et al. (1989), Schatzman (1990), Zahn (1987, 1990, 1992), Maeder & Zahn (1998) and Elliott, Miesh & Toomre (2000). The angular momentum conservation, equations (8b) involves the Reynolds stresses R_{ij} which are given by equations (15a). If one is interested in the convective zone (CZ) where there is strong mixing, one can assume that the μ-gradients are rather small, thus considerably simplifying the problem since only the T-fluxes remain. If, however, one studies the transport of angular momentum below the CZ in the stably stratified radiative zone (RZ) where mixing is significantly less efficient, such μ-gradients cannot be neglected. In both cases, the angular momentum equation depends on the meridional currents $\bar{u}_r$ and $\bar{u}_\theta$, for which the dynamic equations are given equations (6e) and (6f). Once again, different treatments are required depending on whether one works in the CZ or RZ. In the latter case, $\bar{u}_r$ and $\bar{u}_\theta$ are usually taken to be of the Eddington–Sweet–Vogt form (e.g. Zahn 1992).

11.2 Solar tachocline

Helioseismological data (Antia, Basu & Chitre 1998; Charbonneau et al. 1999; Corbard, Berthomieu & Provost 1998; Li & Wilson 1998) have shown that solar rotation changes quite abruptly from latitude-dependent in the CZ to (almost) uniform in the RZ. Since thus far most R_{ij} have been expressed as

$$R_{ij} \sim a_{ij} r \partial\Omega/\partial r + b_{ij}\partial\Omega/\partial\theta, \tag{42a}$$

the R_{ij} abruptly vanish in the tachocline and so do the diffusivities. The model presented here is no longer of the form (42a) but rather of the form

$$R_{ij} \sim f(S_{ij}, V_{ij}) + c_{ij}(\Omega + \Omega_0) + d_{ij}(\bar{u}_r, \bar{u}_\theta) + B_{ij}. \tag{42b}$$

In (42b) there are four new terms: a vorticity term V_{ij}, a linear dependence on $\Omega(r,\theta)$ and Ω_0, a term d_{ij} that depends on the meridional flow $\bar{u}_r$, $\bar{u}_\theta$ and a buoyancy term B_{ij}. The last two terms are missing in all previous models, while the Ω-term was introduced by Rudiger (1989) using heuristic arguments. Here, the full form (42b) is derived from the Navier–Stokes equations.

Because of the abrupt change in the functional dependence of Ω from the CZ to the RZ, the layer between the two is the seat of strong vertical shear. Since radiative losses become increasingly important and since they weaken the temperature gradients and thus the negative effects of stratification, the extent to which turbulence 'survives' is a delicate matter that linear stability analysis cannot resolve (Schatzman et al. 2000; Dikpati & Gilman 2001). Coming down from the CZ where the mixing is the strongest and the radiative losses the weakest, into the RZ where the opposite is true, one needs to follow 'the path toward the fading of turbulence' by tracking the behaviour of turbulent kinetic energy E as a function of two factors that represent opposite effects: the Richardson number which quantifies the strength of shear (a source) versus stratification (a sink), and the Peclet number Pe which characterizes the strength of radiative losses. The difficulty is that Pe itself is a function of the dynamical variable E and ϵ, equation (38d), and thus Pe cannot be assigned a fixed value since it is getting smaller as one enters the RZ. Thus Pe is part of the solution rather than a pre-fixed value.

Our approach, which can be seen as being top-down since it follows how turbulence weakens in the presence of radiative losses (Canuto 1998), indicates that turbulence persists longer than expected on linear stability grounds. With the present more general formalism, the problem ought to be revisited since the presence of meridional currents can further alter the conclusions of recent studies (Schatzman et al. 2000).

11.3 Mixing and advecting

In general, the diffusivities that we have derived are represented by non-symmetric tensors. We have shown that the symmetric part gives rise to mixing (diffusion), while the antisymmetric part gives rise to advection (also called stirring, folding or streaking). Thus the common assumption, that turbulence gives rise only to mixing while advection is entirely due to a mean flow, is proven to be incorrect.

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